Journal of Engineering Mathematics **28**: 117–127, 1994.

Reissner–Sagoci problem for a non-homogeneous half-space with surface constraint

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Received 30 June 1992; accepted in revised form 30 June 1993

Abstract. This paper deals with the problem of twisting of a non-homogeneous, isotropic, half-space by rotating a circular part of its boundary surface ($0 \leq r \leq a$, $z = 0$) through a given angle. A ring ($a < r < b$, $z = 0$) outside this circle is stress-free and the remaining part ($r > b$, $z = 0$) is rigidly clamped. The shear modulus μ is assumed to vary with the cylindrical coordinates r, z by the power law ($\mu = \mu_{\alpha,\beta} r^{\alpha} z^{\beta}$). Such a dependence is of practical interest in the context of Soil Mechanics. The problem leads to a Fredholm integral equation of the second kind which is solved numerically, giving an evaluation of the influence of non-homogeneity on the torque at the surface and the stress intensity factor. The homogeneous case studied in [16] is recovered. Expressions for some quantities of physical importance such as the torque applied at the surface and stress intensity factor are obtained. It appears from our investigation that the influence of clamping dies out with increasing α and β. Quantitative evaluations are given and some curves for the relative increase, due to clamping, in the torque and in the stress intensity factor are presented.

1. Introduction

The standard Reissner–Sagoci problem [1]–[5] is that of determining the components of stress and displacement in the interior of the semi-infinite homogeneous isotropic solid ($z > 0$), when a circular area ($0 \leq r \leq a$) of its boundary surface ($z = 0$) is forced to rotate through an angle, γ, about the z-axis. It is assumed that the part of the boundary surface which lies outside this circle is stress-free. This problem for a non-homogeneous half-space or large thick plate, which is of some practical importance, was considered by several authors [6]–[15].

Recently, Singh et al. [16] considered the Reissner–Sagoci problem for a homogeneous half-space with a surface constraint that the ring ($a < r < b$, $z = 0$) is stress free and the remainder part ($r > b$, $z = 0$) is rigidly clamped.

This paper is to consider the constrained problem for a non-homogeneous isotropic half-space with shear modulus in the form $\mu_{\alpha,\beta} r^{\alpha} z^{\beta}$, where $\alpha > -2$, $0 \leq \beta < 1$ and $\mu_{\alpha,\beta}$-constant. This kind of dependence occurs in some soil materials.

The problem leads to a system of triple integral equations which is further reduced to a single integral equation of Fredholm type of the second kind for an auxiliary function.

This Fredholm integral equation for the general values of the parameters α, β and $\varepsilon = a/b$ may be solved numerically. Moreover one can obtain an iterative solution in the form of a convergent power series in the ratio ε.

Expressions for some quantities of physical importance such as the torque applied at the surface and stress intensity factor at the rim ($z = 0, r = a - 0$) are given in terms of the auxiliary function.

Some curves are presented for the relative increase in the torque and in the stress intensity factor due to clamping against the ratio ε for different values α and β.

It appears from our investigations that the influence of clamping dies out with increasing α and β. This may be clearly remarked on the figures.

2. Formulation of the problem

We consider the torsion of a non-homogeneous, semi-infinite solid ($z \geq 0$) by rotating the circular area ($0 \leq r \leq a$) of its bounding surface ($z = 0$) through an angle γ about the z-axis. The part ($z = 0$, $b < r < \infty$) is assumed to be rigidly clamped. The remaining part ($a < r < b$) of this surface is stress free (Fig. 1). Moreover, it is assumed that the solid is clamped at infinity.

We take the shear modulus of the solid in the form:

$$\mu = \mu_{\alpha,\beta} r^{\alpha} z^{\beta} \tag{1}$$

where $\alpha > -2$, $0 \leq \beta < 1$, $\mu_{\alpha,\beta}$-constant.

For the axisymmetrical torsion problems, the only non vanishing displacement is the circumferential one, u_θ. The non vanishing stress components $\tau_{\theta r}$, $\tau_{\theta z}$ are related to u_θ through the relations:

$$\tau_{\theta r} = \mu r \frac{\partial}{\partial r}(u_\theta / r) \, ; \qquad \tau_{\theta z} = \mu \frac{\partial}{\partial z}(u_\theta) \, . \tag{2}$$

The only nontrivially satisfied equilibrium equation is:

$$\frac{\partial \tau_{\theta r}}{\partial r} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2}{r} \tau_{\theta r} = 0 \, . \tag{3}$$

Substituting (2) into (3) and taking (1) in account, we obtain the following partial differential equation for determining the displacement component u_θ:

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{(1+\alpha)}{r} \frac{\partial u_\theta}{\partial r} - \frac{(1+\alpha)}{r^2} u_\theta + \frac{\beta}{z} \frac{\partial u_\theta}{\partial z} + \frac{\partial^2 u_\theta}{\partial z^2} = 0 \, . \tag{4}$$

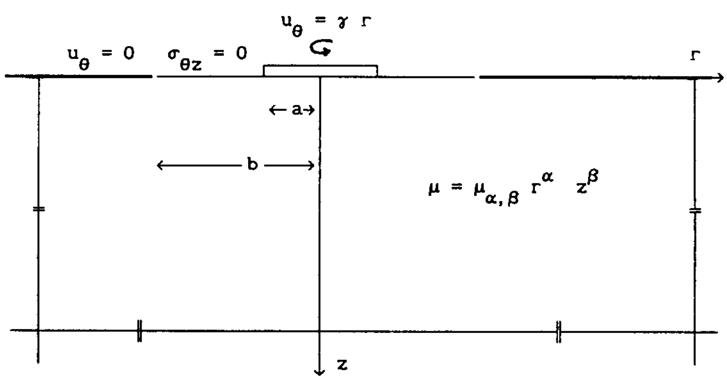

Fig. 1. Formulation of the problem.

The boundary conditions of the problem are:

$$u_\theta(r,0)=\gamma r\,,\quad (0\leq r\leq a)\,, \tag{5}$$

$$\tau_{\theta z}(r,0)=0\,,\quad (a<r<b)\,, \tag{6}$$

$$u_\theta(r,0)=0\,,\quad (r\geq b)\,. \tag{7}$$

Also,

$$u_\theta,\ \tau_{\theta 4} \text{ and } \tau_{\theta z}\to 0 \quad \text{as } r^2+z^2\to\infty\,. \tag{8}$$

3. Reduction to an integral equation

The solution of equation (4) which satisfies condition (8) takes the form:

$$u_\theta(r,z)=r^{1-\nu}z^p\int_0^\infty \lambda^p A(\lambda)J_\nu(\lambda r)K_p(\lambda z)\,\mathrm{d}\lambda\,, \tag{9}$$

where $p=(1-\beta)/2$, $\nu=1+(\alpha/2)$, $J_\nu(x)$ denotes the Bessel function of the first kind, $K_p(x)$ is the modified Bessel function of the second kind and $A(\lambda)$ is an unknown function to be found.

The stress components corresponding to this displacement may be obtained by substituting (9) into (2) to get

$$\tau_{\theta z}(r,z)=-\mu_{\alpha,\beta}r^{\nu-1}z^{1-p}\int_0^\infty \lambda^{1+p}A(\lambda)J_\nu(\lambda r)K_{p-1}(\lambda z)\,\mathrm{d}\lambda\,, \tag{10}$$

$$\tau_{\theta r}(r,z)=-\mu_{\alpha,\beta}r^{\nu-1}z^{1-p}\int_0^\infty \lambda^{1+p}A(\lambda)J_{\nu+1}(\lambda r)K_p(\lambda z)\,\mathrm{d}\lambda\,. \tag{11}$$

The boundary conditions (5)–(7) reduce to the following system of triple integral equations:

$$\int_0^\infty A(\lambda)J_\nu(\lambda r)\,\mathrm{d}\lambda=\frac{2^{1-p}}{\Gamma(p)}\gamma r^\nu\,,\quad (0\leq r\leq a)\,, \tag{12}$$

$$\int_0^\infty \lambda^{2p}A(\lambda)J_\nu(\lambda r)\,\mathrm{d}\lambda=0\,,\quad (a<r<b)\,, \tag{13}$$

$$\int_0^\infty A(\lambda)J_\nu(\lambda r)\,\mathrm{d}\lambda=0\,,\quad (b\leq r<\infty)\,. \tag{14}$$

These may be solved by assuming [16]

$$\int_0^\infty \lambda^{2p}A(\lambda)J_\nu(\lambda r)\,\mathrm{d}\lambda=\begin{cases} f_1(r)\,, & (0\leq r\leq a)\,,\\ f_2(r)\,, & (b\leq r<\infty)\,.\end{cases} \tag{15}$$

By the Hankel inversion theorem, equations (13) and (15) give

$$\lambda^{2p-1}A(\lambda) = \int_0^a xf_1(x)J_\nu(\lambda x)\,dx + \int_b^\infty xf_2(x)J_\nu(\lambda x)\,dx\,. \tag{16}$$

Substituting the value $A(\lambda)$ from (16) into equations (12) and (14), we get

$$\int_0^a xf_1(x)L(x,r)\,dx + \int_b^\infty xf_2(x)L(x,r)\,dx = \begin{cases} \dfrac{2^{1-p}}{\Gamma(p)}\gamma r^\nu\,, & (0 \leqslant r \leqslant a)\,,\\ 0\,, & (b \leqslant r < \infty)\,, \end{cases} \tag{17}$$

where

$$L(x,r) = \int_0^\infty \lambda^{1-2p}J_\nu(\lambda x)J_\nu(\lambda r)\,d\lambda\,. \tag{18}$$

Making use of the result in Appendix, write

$$L(x,r) = \frac{2^{2-2p}}{\{\Gamma(p)\}^2}(xr)^{-\nu}\int_0^{\min(x,r)} \frac{s^{1+2\nu-2p}}{(x^2-s^2)^{1-p}(r^2-s^2)^{1-p}}\,ds$$

or (19)

$$= \frac{2^{2-2p}}{\{\Gamma(p)\}^2}(xr)^{\nu}\int_{\max(x,r)}^\infty \frac{s^{1-2\nu-2p}}{(s^2-x^2)^{1-p}(s^2-r^2)^{1-p}}\,ds\,.$$

Substituting (19) into (17) and interchanging the order of integration, one obtains

$$\int_0^r \frac{s^{1+2\nu-2p}F_1(s)}{(r^2-s^2)^{1-p}}\,ds = 2^{p-1}\Gamma(p)\gamma r^{2\nu} - r^{2\nu}\int_b^\infty \frac{s^{1-2\nu-2p}F_2(s)}{(s^2-r^2)^{1-p}}\,ds\,, \quad (0 \leqslant r \leqslant a)\,, \tag{20}$$

$$\int_r^\infty \frac{s^{1-2\nu-2p}F_2(s)}{(s^2-r^2)^{1-p}}\,ds = -r^{-2\nu}\int_0^a \frac{s^{1+2\nu-2p}F_1(s)}{(r^2-s^2)^{1-p}}\,ds\,, \quad (b \leqslant r < \infty)\,. \tag{21}$$

where

$$\left.\begin{aligned} F_1(s) &= \int_s^a \frac{x^{1-\nu}f_1(x)}{(x^2-s^2)^{1-p}}\,dx\,, \quad (0<s<a)\\ F_2(s) &= \int_b^s \frac{x^{1+\nu}f_2(x)}{(s^2-x^2)^{1-p}}\,dx\,. \quad (b<s<\infty) \end{aligned}\right\}. \tag{22}$$

Regarding the right-hand sides of equations (20) and (21) as known functions of r, these equations are of an Abel type. Hence their solutions are

$$s^\nu F_1(s) = \frac{2\Gamma(1+\nu)}{\Gamma(p)\Gamma(1+\nu-p)}\Big\{2^{p-1}\Gamma(p)\gamma s^\nu - \int_b^\infty \lambda^{-1}(s/\lambda)^\nu\lambda^{-\nu}F_2(\lambda)\,{}_2F_1(1-p,1+\nu;1+\nu-p;(s/\lambda)^2)\,d\lambda\Big\}\,, \quad (0 \leqslant s \leqslant a)\,, \tag{23}$$

$$s^{-\nu}F_2(s) = \frac{-2\Gamma(1+\nu)}{\Gamma(p)\Gamma(1+\nu-p)}\left(\int_0^a \lambda^{-1}(\lambda/s)^{2+\nu-2p}\lambda^{\nu}F_1(\lambda)\right.$$
$$\left.\times\, {}_2F_1(1-p, 1+\nu; 1+\nu-p; (\lambda/s)^2)\,\mathrm{d}\lambda\right), \quad (b<s<\infty), \tag{24}$$

where ${}_2F_1$ denotes the hypergeometric function.

In obtaining (23) and (24), we have made use of the following integrals:

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_0^s \frac{r^{1+2\nu}}{(s^2-r^2)^p}\,\mathrm{d}r = \frac{\Gamma(1+\nu)\Gamma(1-p)}{\Gamma(1+\nu-p)}\,s^{1+2\nu-2p}, \tag{25}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_0^s \frac{r^{1+2\nu}}{(s^2-r^2)^p(\lambda^2-r^2)^{1-p}}\,\mathrm{d}r = \left(\frac{\Gamma(1+\nu)\Gamma(1-p)}{\Gamma(1+\nu-p)}\right)\lambda^{2p-2}s^{1+2\nu-2p}$$
$$\times\, {}_2F_1(1-p, 1+\nu; 1+\nu-p; (s/\lambda)^2), \quad (\lambda>s), \tag{26}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_s^\infty \frac{r^{1-2\nu}}{(r^2-s^2)^p(r^2-\lambda^2)^{1-p}}\,\mathrm{d}r = -\left(\frac{\Gamma(1+\nu)\Gamma(1-p)}{\Gamma(1+\nu-p)}\right)s^{-1-2\nu}$$
$$\times\, {}_2F_1(1-p, 1+\nu; 1+\nu-p; (\lambda/s)^2), \quad (\lambda<s). \tag{27}$$

In order to reduce equations (23) and (24) to a convenient form, set

$$\begin{aligned} S^{\nu}F_1(s) &= 2^p\frac{\Gamma(1+\nu)}{\Gamma(1+\nu-p)}\,\gamma a^{(\nu-p+1/2)}s^{p-1/2}\kappa_1(s), \\ \lambda^{-\nu}F_2(\lambda) &= 2^p\frac{\Gamma(1+\nu)}{\Gamma(1+\nu-p)}\,\gamma a^{(\nu-p+1/2)}\lambda^{p-1/2}\kappa_2(\lambda), \end{aligned} \tag{28}$$

and perform a change of variables $s=au$, $\lambda=bv$, in equation (23) and $s=bv$, $\lambda=aw$, in equation (24), as a consequence of which these equations assume the form

$$\kappa_1(au) = u^{\nu-p+1/2} - \frac{2\Gamma(1+\nu)}{\Gamma(p)\Gamma(1+v+p)}\,\varepsilon^{(\nu-p+1/2)}\int_1^\infty v^{-1}(u/v)^{\nu-p+1/2}\kappa_2(bv)$$
$$\times\, {}_2F_1(1-p, 1+\nu; 1+\nu-p; (u/v)^2\varepsilon^2)\,\mathrm{d}v, \quad (0\leq u<1), \tag{29}$$

$$\kappa_2(bv) = \frac{-2\Gamma(1+\nu)}{\Gamma(p)\Gamma(1+\nu-p)}\,\varepsilon^{(\nu-p+3/2)}\int_0^1 w^{-1}(w/v)^{\nu-p+3/2}\kappa_1(aw)$$
$$\times\, {}_2F_1(1-p, 1+\nu; 1+\nu-p; (w/v)^2\varepsilon^2)\,\mathrm{d}w, \quad (1<v<\infty), \tag{30}$$

where $\varepsilon = a/b$.

Substituting for $\kappa_2(bv)$ from (30) into (29), the following Fredholm integral equation of the second kind is obtained for $\kappa_1(au)$:

$$\kappa_1(au) = u^{\nu-p+1/2} + \int_0^1 K(u,\lambda)\kappa_1(a\lambda)\,\mathrm{d}\lambda, \quad (0<u<1), \tag{31}$$

with the symmetric kernel

$$K(u, \lambda) = \varepsilon^{2(1+\nu-p)}(\lambda u)^{\nu-p+1/2}\left[\frac{2\Gamma(1+\nu)}{\Gamma(p)\Gamma(1+\nu-p)}\right]^2 \int_0^1 w^{1+\nu-2p} \times {}_2F_1(1-p, 1+\nu; 1+\nu-p; u^2w^2\varepsilon^2)\,{}_2F_1(1-p, 1+\nu; 1+\nu-p; \lambda^2w^2\varepsilon^2)\,\mathrm{d}w\,. \tag{32}$$

4. Expressions for some physical quantities

The shear component $\tau_{\theta z}$ inside the circle ($z=0,\ 0\leq r\leq a$) is found to be

$$\tau_{\theta z}(r,0) = -\mu_{\alpha,\beta}2^{-p}\Gamma(1-p)r^{\nu-1}f_1(r)\,, \quad (0\leq r\leq a)\,. \tag{33}$$

From equation (22), which is of Abel type, we get

$$f_1(r) = \frac{-2}{\Gamma(p)\Gamma(1-p)}\, r^{\nu-1}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{sF_1(s)}{(s^2-r^2)^p}\,\mathrm{d}s\,, \tag{34}$$

$$\tau_{\theta z}(r,0) = \mu_{\alpha,\beta}\frac{2^{1-p}}{\Gamma(p)}\, r^{2(\nu-1)}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{sF_1(s)}{r^{(s^2-r^2)^p}}\,\mathrm{d}s\,. \tag{35}$$

The torque T required to produce the rotation is:

$$T = -2\pi\int_0^a r^2\tau_{\theta z}(r,0)\,\mathrm{d}r\,. \tag{36}$$

Substituting (35) into (36), one obtains

$$T = \mu_{\alpha,\beta}\gamma a^{2(1+\nu-p)}\frac{4\pi\Gamma(1-p)}{\Gamma(p)}\left[\frac{\Gamma(1+\nu)}{\Gamma(1+\nu-p)}\right]^2\int_0^1 u^{\nu-p+1/2}\kappa_1(au)\,\mathrm{d}u\,. \tag{37}$$

The second quantity most interesting for applications is the stress intensity factor

$$K_t = \lim_{r\to a-0}\,[a-r]^p\tau_{\theta z}(r,0)\,,$$

which may be derived from (35) and (28). It is connected with the main auxiliary function κ_1 by the relation

$$K_t = -\mu_{\alpha,\beta}\frac{2^{1-p}\Gamma(1+\nu)}{\Gamma(p)\Gamma(1+\nu-p)}\,\gamma a^{2\nu-p-1}\kappa_1(a)\,. \tag{38}$$

It is important to note that, if the ratio ($\varepsilon = a/b$) tends to zero (i.e. b tends to infinity), the kernel $K(u,t)$ tends to zero, and we recover the exact solution

$$\kappa_1^\infty(au) = u^{\nu-p+1/2}\,. \tag{39}$$

This limiting case corresponds to the unconstrained Reissner–Sagoci problem for a non-homogeneous half-space studied by Singh [10]. If we denote the torque and the stress intensity factor in this case by T^∞ and K_t^∞ respectively, then

$$T^{\infty}=\mu_{\alpha,\beta}\gamma a^{2(1+\nu-p)}\frac{2\pi\Gamma(1-p)}{(1+\nu-p)\Gamma(p)}\left[\frac{\Gamma(1+\nu)}{\Gamma(1+\nu-p)}\right]^{2}, \tag{40}$$

$$K_t^{\infty}=-\mu_{\alpha,\beta}\frac{2^{1-p}\Gamma(1+\nu)}{\Gamma(p)\Gamma(1+\nu-p)}\gamma a^{2\nu-p-1}. \tag{41}$$

Using equations (37), (38), (40) and (41), we get the relative increase in the torque T_p and in the stress intensity factor K_p due to clamping:

$$T_p=\frac{T-T^{\infty}}{T^{\infty}}=\left[2(1+\nu-p)\int_0^1 u^{\nu-p+1/2}\kappa_1(au)\,\mathrm{d}u-1\right], \tag{42}$$

$$K_p=\frac{K_t-K_t^{\infty}}{K_t^{\infty}}=[\kappa_1(a)-1]. \tag{43}$$

The third quantity is the displacement component $u_\theta(r,0)$ in the ring ($z=0$, $a<r<b$). Its expression in terms of the two functions κ_1, κ_2 is found to be

$$u_\theta(r,0)=\frac{2\Gamma(1+\nu)\gamma a^{\nu-p+1/2}}{\Gamma(p)\Gamma(1+\nu-p)}r^{1-2\nu}\left\{\int_0^a\frac{s^{\nu-p+1/2}\kappa_1(s)}{(r^2-s^2)^{1-p}}\,\mathrm{d}s+r^{2\nu}\int_b^{\infty}\frac{s^{-\nu-p+1/2}\kappa_2(s)}{(s^2-r^2)^{1-p}}\,\mathrm{d}s\right\},\quad (a<r<b). \tag{44}$$

For the unconstrained case ($b\to\infty$), this component takes the form:

$$u_\theta^{\infty}(r,0)=\frac{\Gamma(1+\nu)}{\Gamma(p)\Gamma(1+\nu-p)}\gamma r B_{(a^2/r^2)}(\nu+1-p,p), \tag{45}$$

where $B_{(x)}(m,n)$ is the incomplete beta function. For $\nu=1$ (i.e. $\alpha=0$), form (45) agrees with the result of Kassir [8].

5. Solution of the integral equation (31)

Equation (31) may be solved numerically for general values of the parameters p, ν, ε. Following Kantorovich and Krylov [18], this integral equation is replaced by a finite system of linear algebraic equations. We use values $\nu=0.25$, 0.5, 0.75, 1, 1.5, 2, 2.5, $p=0.1$, 0.2, 0.3, 0.4, 0.5, and $\varepsilon=0.1$, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, for the parameters. Some results are shown graphically on Figs. 2–7, where we plotted the variations of T_p and K_p with ε.

On the other hand, when the ratio ε is sufficiently small, an iterative solution of equation (31) may be obtained as a convergent power series in ε. To derive this solution, expand the kernel $K(u,\lambda)$ in powers of ε to yield

$$K(u,\lambda)=\sum_{m=0}^{\infty}\phi_m(u,\lambda)\varepsilon^{2m+2+n}, \tag{46}$$

where $n=2(\nu-p)$,

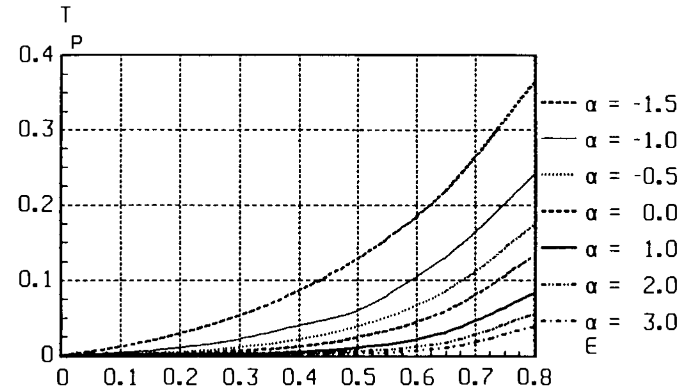

Fig. 2. The values of T_p for $\beta = 0.0$.

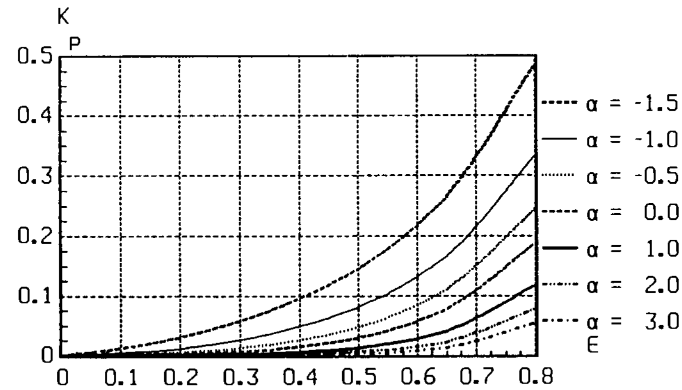

Fig. 3. The values of K_p for $\beta = 0.0$

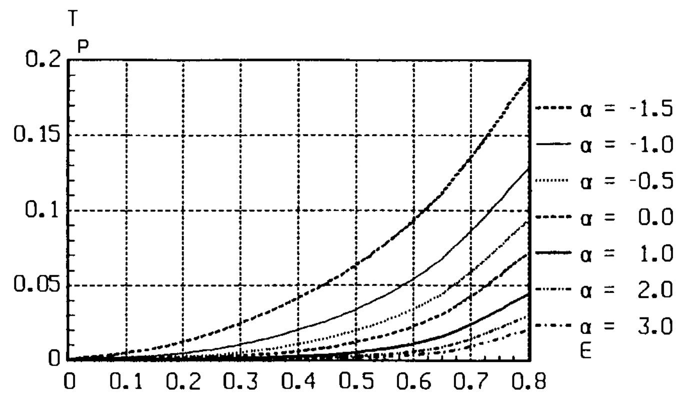

Fig. 4. Values of T_p for $\beta = 0.2$.

$$\phi_m(u,\lambda) = 2\left[\frac{\Gamma(1+\nu)}{\Gamma(p)\Gamma(1+\nu-p)}\right]^2 \left(\frac{(\lambda u)^{(1+n)/2}}{(1+\nu+m-p)}\right) \sum_{k=0}^{m} \left(\frac{(1-p)_k(1+\nu)_k u^{2k}}{(1+\nu-p)_k k!} \times \frac{(1-p)_{m-k}(1+\nu)_{m-k}\lambda^{2(m-k)}}{(1+\nu-p)_{m-k}(m-k)!}\right) \tag{47}$$

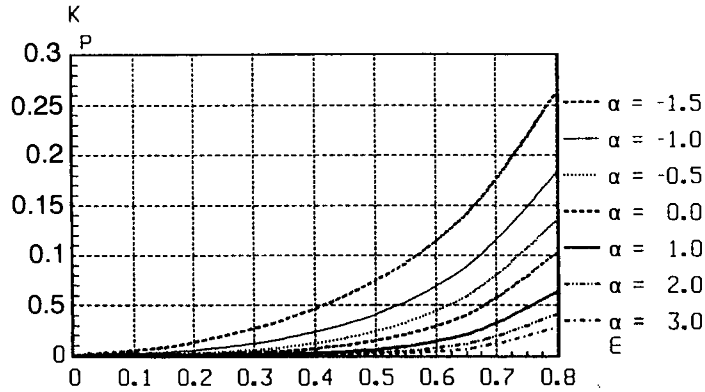

Fig. 5. The values of K_p for $\beta = 0.2$.

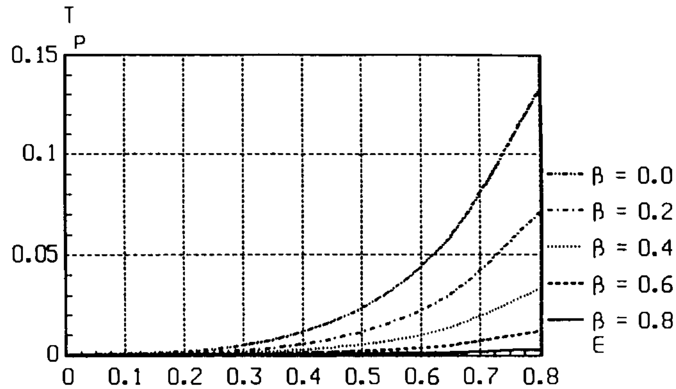

Fig. 6. Values of T_p for $\alpha = 0$.

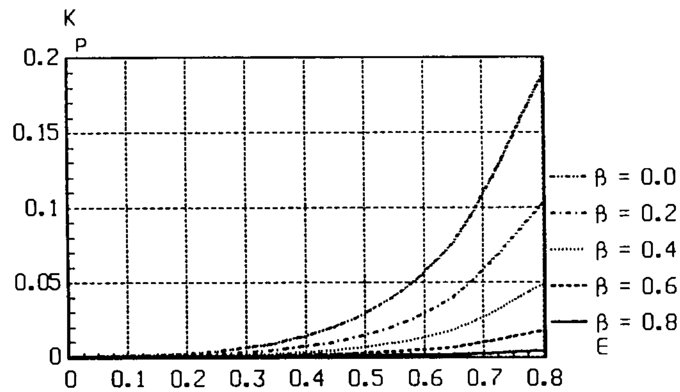

Fig. 7. Values of K_p for $\alpha = 0$.

and $(a)_n$ denotes the shifted factorial defined as

$$(a)_n = \Gamma(a+n)/\Gamma(a) .$$

The iterative solution is found and the three-terms expansions for the quantities T_p and K_p are then computed:

$$
\begin{aligned}
T_p^{(n)} &= \left(\frac{\Gamma(1+\nu)}{\Gamma(p)\Gamma(2+n/2)}\right)^2 \left\{\varepsilon^{2+n} + \frac{2+n}{(2+n/2)^2}(1-p)(1+\nu)\varepsilon^{4+n} \right. \\
&\qquad \left. + \left(\frac{\Gamma(1+\nu)}{\Gamma(p)\Gamma(2+n/2)}\right)^2 \varepsilon^{4+2n} + O(\varepsilon^q)\right\}, \\
K_p^{(n)} &= \left(\frac{\Gamma(1+\nu)}{\Gamma(p)\Gamma(2+n/2)}\right)^2 \left\{\varepsilon^{2+n} + \frac{3+n}{(2+n/2)^2}(1-p)(1+\nu)\varepsilon^{4+n} \right. \\
&\qquad \left. + \left(\frac{\Gamma(1+\nu)}{\Gamma(p)\Gamma(2+n/2)}\right)^2 \varepsilon^{4+2n} + O(\varepsilon^q)\right\},
\end{aligned}
\tag{48}
$$

where $n = 2(\nu - p) = \alpha + \beta + 1 > -1$ and $q = \min(6+n, 6+2n, 6+3n)$.

The obtained results are in full agreement with those of the former method.

6. Conclusions

On the basis of our numerical calculations, we arrive at the following conclusions.

(i) The influence of clamping dies out with increasing α and β.

(ii) For any values of β and for all non-negative values of α, the influence of constraint may be neglected, since it induces relative variations in the torque and coefficient of concentration less than 2% as long as ε does not exceed 0.4. This also holds for negative values of α, provided β is greater than 0.4.

Appendix

Following Erdélyi et al. [17, form. 19.3 (1)], if $x < r$

$$
\begin{aligned}
L(x,r) = \int_0^\infty t^{1-2p} J_\nu(tx) J_\nu(tr)\,\mathrm{d}t = \Bigg(&\frac{2^{1-2p}\Gamma(1+\nu-p)}{\Gamma(p)\Gamma(1+\nu)} x^\nu r^{2p-\nu-2} \\
&\times {}_2F_1(1-p,\ 1+\nu-p;\ \nu+1;\ (x/r)^2)\Bigg).
\end{aligned}
$$

Using Euler's integral representation of the hypergeometric function ${}_2F_1$, one obtains

$$
L(x,r) = \frac{2^{1-2p}}{\{\Gamma(p)\}^2} \frac{x^\nu}{r^{2+\nu-2p}} \int_0^1 t^{\nu-p}(1-t)^{p-1}(1-tx^2/r^2)^{p-1}\,\mathrm{d}t. \tag{A.1}
$$

Substituting $t = s^2/x^2$ in (A.1), we get

$$
L(x,r) = \frac{2^{2-2p}}{\{\Gamma(p)\}^2}(xr)^{-\nu} \int_0^x \frac{s^{1+2\nu-2p}}{(x^2-s^2)^{1-p}(r^2-s^2)^{1-p}}\,\mathrm{d}s. \tag{A.2}
$$

Substituting $t = r^2/s^2$ in (A.1), we get

$$
L(x,r) = \frac{2^{2-2p}}{\{\Gamma(p)\}^2}(xr)^{\nu} \int_r^\infty \frac{s^{1-2\nu-2p}}{(s^2-x^2)^{1-p}(s^2-r^2)^{1-p}}\,\mathrm{d}s. \tag{A.3}
$$

Acknowledgements

One of the referees is gratefully acknowledged for his careful reading and useful suggestions.

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